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Reduction of observer order by differentiation, almost controllability subspace covers and minimal order PID-observers

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This note generalizes the geometric theory around minimal and reduced order observers to the situation in which differentiation of certain components of the observed output is allowed. A geometric theory involving the notion of PID-observer is introduced, using the concept of almost complementary observability subspace. A generalization of the notions of dynamic cover and cover index is defined in the context of almost controllability subspaces. We prove a result on the existence of minimal dimension covers for one-dimensional subspaces. These ideas are used to define the concept of minimal order PID-observer and to establish the existence of minimal order PID-observers for a single linear functional of the state.

Keywords: Geometric theory, PID-observers, Almost controllability subspaces, Cover index, Almost complementary observability subspaces.

1. Introduction

In this note we will consider the dynamical system

$$\Sigma: \quad \dot{x} = Ax, \quad y = Cx, \tag{1.1}$$

with $x \in \mathscr{X} := \mathbb{R}^n$, $y \in \mathscr{Y} := \mathbb{R}^p$ and A and C linear maps (matrices of appropriate dimensions). In Σ, we will interpret y as an *observed output*. It will be assumed throughout that C is subjective and that the pair (C, A) is observable.

It is well known that a 'full order' dynamic observer for the state of the system Σ can be found as follows [1,2]: let Λ be a symmetric set (i.e. non-real elements of Λ appear in conjugate pairs) of n complex numbers. Let $K: \mathscr{Y} \to \mathscr{Y}$ be a map such that $\sigma(A + KC) = \Lambda$ and consider the n-th order observer

$$\Sigma_{\text{obs}}: \quad \dot{w} = Jw - Ky, \tag{1.2a}$$

$$\hat{x} = w, \tag{1.2b}$$

with $J := A + KC$. Obviously, the error $e := \hat{x} - x$ satisfies $\dot{e} = Je$. Taking $\sigma(J) = \Lambda$ in the open left half complex plane then yields $e(t) \to 0$ $(t \to \infty)$ and hence $\hat{x}(t)$ *ultimately* identifies $x(t)$. As pointed out in [2] and [3], the dynamic order of the above observer is unnecessarily large, since from the observation $y(t)$ it is possible at once to recover the part of the state vector modulo ker C. In this way it is possible to reduce the order of the observer dynamics to $n -$ codim ker C. In [3] the existence of this reduced order observer was established using the dual version of the following proposition:

Proposition 1.1 [7]. *Assume that (C, A) is observable. Let Λ be a symmetric set of $n - p$ complex numbers. Then there exists a subspace $\mathscr{S} \subset \mathscr{X}$ and a linear map $K: \mathscr{Y} \to \mathscr{X}$ such that*

$$\ker C \oplus \mathscr{S} = \mathscr{X}, \tag{1.3}$$

$$(A + KC)\mathscr{S} \subset \mathscr{S} \tag{1.4}$$

and

$$\sigma((A + KC) \bmod \mathscr{S}) = \lambda. \quad \square \tag{1.5}$$

We will briefly recall the construction leading the the 'reduced order' observer. Let $V: \mathscr{S} \to \mathbb{R}^{n-p}$ be a map such that $\mathscr{S} = \ker V$. From (1.3) we have

$$\ker\begin{pmatrix} C \\ V \end{pmatrix} = \{0\}.$$

Hence there are matrices M and N of appropriate

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dimensions such that

$$MC + NV = I_{n \times n}. \tag{1.6}$$

Let $K: \mathscr{Y} \to \mathscr{X}$ be as in Proposition 1.1. Clearly, (1.4) and (1.5) are equivalent to the existence of a map $J: \mathbb{R}^{n-p} \to \mathbb{R}^{n-p}$ with the property that

$$V(A + KC) + JV \quad \text{and} \quad \sigma(J) = \Lambda.$$

Consider now the $(n-p)$-th order observer

$$\Sigma_{\text{obs}}: \quad \dot{w} = Jw - VKy, \tag{1.7a}$$

$$\hat{x} = Nw + My, \tag{1.7b}$$

and define $e := w - Vx$. Obviously $\dot{e} = Je$. Taking $\sigma(J) = \Lambda$ in the open left half complex plane yields $e(t) \to 0$ $(t \to \infty)$. From (1.3), note that every $x \in \mathscr{X}$ can be written uniquely as $x = x_1 \oplus x_2$, with $x_1 \in \ker C$ and $x_2 \in \mathscr{S} = \ker V$. From (1.6),

$$x_1 = NVx_1 = NVx \quad \text{and} \quad x_2 = MCx_2 = MCx.$$

Therefore, $Nw(t) \to x_1(t)$ $(t \to \infty)$ and $My(t) = x_2(t)$. It follows that

$$\hat{x}(t) - x(t) = Ne(t) \to 0 \quad (t \to \infty),$$

that the state component in $\ker C$ is *ultimately* identified by $Nw(t)$ and that the state component in $\mathscr{S}$ is identified *instantaneously* by $My(t)$. It is vital to note here that this instantaneous identification property is provided by the presence of a *direct feedthrough term* in (1.7b) which is not present in (1.2b). In this sense, the observer (1.7) could be called a *PI-observer* (proportional/integral), while the observer (1.2) could be called an *I-observer* (integral). In this note we will show that it is possible to reduce the dynamic order of the observer even more by allowing direct feedthrough of derivatives of the observation $y(t)$, i.e. by allowing the observer to be a *PID-observer* (differentiation). In Section 3 we will introduce generalizations of the notions of dynamic cover and cover index [4] and subsequently establish a generalization of a result by Wonham and Morse on the existence of minimal dimension covers for one-dimensional subspaces. Finally, in Section 4, we will introduce formal definitions of 'PID-observer' and 'PID-observer index'. The results from Section 3 will be dualized to establish the existence of minimal order PID-observers for a single linear functional of the state.

2. Reduction of dynamic order by differentiation

Instead of assuming the *entire* observation $y(t)$, together with *all* its derivatives $y^{(1)}(t)$, $y^{(2)}(t), \ldots$ to be available for instantaneous identification of the plant state, we will take the following option. Let

$$y(t) = \left(y_1(t), \ldots, y_p(t)\right)^{\mathrm{T}}.$$

For each component $y_i(t)$, specify an integer κ_i such that $-1 \leqslant \kappa_i \leqslant n-1$. $\kappa_i \geqslant 0$ will mean that $y_i(t)$, $y_i^{(1)}(t), \ldots, y_i^{(\kappa_i)}(t)$ may be used for direct feedthrough. $\kappa_i = -1$ will mean that neither $y_i(t)$ nor any of its derivatives may used for direct feedthrough. In a suitable basis it can be arranged that $\kappa_1 \geqslant \kappa_1 \geqslant \cdots \geqslant \kappa_p$. Define integers ν_i $(i = 0, \ldots, \kappa_1)$ by

$$\nu_i := \text{the number of integers in the set } \{\kappa_1, \ldots, \kappa_p\} \text{ which are } \geqslant i. \tag{2.1}$$

Clearly, $\nu_0 \geqslant \nu_1 \geqslant \cdots \geqslant \nu_{\kappa_1}$. Moreover, the above is equivalent to saying that the subvector

$$\left(y_1(t), \ldots, y_{\nu_i}(t)\right)^{\mathrm{T}} \in \mathscr{Y}_i := \mathbb{R}^{\nu_i}$$

may be used for direct feedthrough, together with all its derivatives up to the order i. More concretely, let $L_i: \mathscr{Y} \to \mathscr{Y}_i$ be linear maps such that

$$\left(y_1, \ldots, y_{\nu_i}\right)^{\mathrm{T}} = L_i y.$$

Then the output equations of our observer should be of the form

$$\hat{x}(t) = F\Big(w(t), L_0 y(t), L_1 y^{(1)}(t), \ldots, L_{\kappa_1} y^{(\kappa_1)}(t)\Big) \tag{2.2a}$$

for some linear map

$$F: \mathscr{W} \oplus \mathscr{Y}_0 \oplus \cdots \oplus \mathscr{Y}_{\kappa_1} \to \mathscr{X}.$$

Here, $w \in \mathscr{W}$ is the state of the observer, which will be assumed to be driven only by $y(t)$. That is, the dynamic part of our observer will be assumed to be of the form

$$\dot{w}(t) = G(w(t), y(t)) \tag{2.2b}$$

for some linear map $G: \mathscr{W} \oplus \mathscr{Y} \to \mathscr{W}$.

Obviously,

$$\ker L_0 \subset \ker L_1 \subset \cdots \subset \ker L_{\kappa_1}.$$

Define a map

$$W: \mathscr{X} \to \mathscr{Y}_0 \oplus \mathscr{Y}_1 \oplus \cdots \oplus \mathscr{Y}_{\kappa_1}$$

by

$$Wx := \begin{pmatrix} L_0 C \\ L_1 CA \\ \vdots \\ L_{\kappa_1} CA^{\kappa_1} \end{pmatrix} x. \tag{2.3}$$

Denote $\mathscr{K}_i := \ker L_i C$. We then have the following inclusion:

$$\ker C =: \mathscr{K} \subset \mathscr{K}_0 \subset \cdots \subset K_{\kappa_1}. \tag{2.4}$$

It follows that $\{\mathscr{K}_i\}$ is *a chain around* $\mathscr{K}$ [5]. Moreover,

$$\ker W = \bigcap_{i=0}^{\kappa_1} A^{-i}\mathscr{K}_i \tag{2.5}$$

from which we obtain that ker W is an *almost complementary observability subspace* with respect to (C, A) [5]. The following theorem is the dual version of [6], Theorem 5.1. It provides a direct generalization of Proposition 1.1.

Theorem 2.1. *Assume that* (C, A) *is observable and suppose that* $\mathscr{N}_a \subset \mathscr{X}$ *is an almost complementary observability subspace. Let* Λ *be a symmetric set of* $\dim \mathscr{N}_a$ *complex numbers. Then there exist a subspace* $\mathscr{S} \subset \mathscr{X}$ *and a map* $K: \mathscr{Y} \to \mathscr{X}$ *such that*

$$\mathscr{N}_a \oplus \mathscr{S} = \mathscr{X}, \tag{2.6}$$

$$(A + KC)\mathscr{S} \subset \mathscr{S} \tag{2.7}$$

and

$$\sigma((A + KC) \bmod \mathscr{S}) = \Lambda. \quad \square \tag{2.8}$$

It will now be shown how this theorem can be applied to reduce the dynamic order of the state observer by allowing direct feedthrough of derivatives of $y(t)$ according to the prespecified integers κ_i. Take $\mathscr{N}_a = \ker W$ in the above theorem. Let Λ be a symmetric set of $n_a := \dim \mathscr{N}_a$ complex numbers. Let $\mathscr{S} \subset \mathscr{X}$ and $K: \mathscr{Y} \to \mathscr{X}$ be as above and let $V: \mathscr{X} \to \mathbb{R}^{n_a}$ be a map such that $\mathscr{S} = \ker V$. From (2.5) we have $\ker\binom{C}{V} = \{0\}$. Therefore, there are maps

$$M: \mathscr{Y}_0 \oplus \mathscr{Y}_1 \oplus \cdots \oplus \mathscr{Y}_{\kappa_1} \to \mathscr{X}$$

and a map $N: \mathbb{R}^{n_a} \to \mathscr{X}$ such that

$$MW + NV = I_{n \times n}. \tag{2.9}$$

Again, (2.7) and (2.8) are equivalent to the existence of a map $J: \mathbb{R}^{n_a} \to \mathbb{R}^{n_a}$ such that

$$V(A + KC) + JV \quad \text{and} \quad \sigma(J) = \Lambda.$$

Let M be partitioned as $M = (M_0, M_1, \ldots, M_{\kappa_1})$. Consider the PID-observer

$$\Sigma_{\text{obs}}: \quad \dot{w} = Jw - VKy, \tag{2.10a}$$

$$\hat{x} = Nw + M_0 L_0 y + M_1 L_1 y^{(1)} + \cdots + M_{\kappa_1} L_{\kappa_1} y^{(\kappa_1)}. \tag{2.10b}$$

Let $e := w - Vx$. As in Section 1, $\dot{e} = Je$ and taking Λ in the open left half complex plane yields $e(t) \to 0$ $(t \to \infty)$. From (2.6), every $x \in \mathscr{X}$ can be written uniquely as $x = x_1 \oplus x_2$ with

$$x_1 \in \mathscr{N}_a = \ker W \quad \text{and} \quad x_2 \in \mathscr{S} = \ker V.$$

Again, from (2.9),

$$x_1 = NVx_1 + Nvx \quad \text{and} \quad x_2 = MWx_2 = MWx.$$

It follows that

$$NW(t) \to x_1(t) \quad \text{as } t \to \infty,$$

that

$$\sum_{i=0}^{\kappa_1} M_i L_i y^{(i)}(t) = x_2(t)$$

and that

$$\hat{x}(t) - x(t) = Ne(t) \to 0 \quad (t \to \infty).$$

Note that the state component in $\mathscr{N}_a$ is ultimately identified from $Nw(t)$, while the state component in $\mathscr{S}$ is identified *instantaneously* from

$$L_0 y(t), L_1 y^{(1)}(t), \ldots, L_{\kappa_1} y^{(\kappa_1)}(t).$$

The dynamic order of the observer (2.10) is $n_a =$ dim ker W. From (2.3) it is clear that

$$\dim \ker W \geqslant \max\left\{0, n - p - \sum_{i=1}^{p} \kappa_i\right\} \tag{2.11}$$

with equality if and only if W has full rank. Therefore, (2.11) only provides a lower bound for the order of the observer (2.10) and this lower bound is achieved if and only if the map W has full rank. One possible way to make sure that W

has full rank is to choose the integers κ_i in the following way: Let $l_1 \geqslant l_2 \geqslant \cdots \geqslant l_p \geqslant 1$ be the observability indices associated with (C, A). Let $c_1, c_2, \ldots, c_p$ be the rows of C. Dualizing a result from [7], Section 5.7, it can be shown that, possibly after relabeling the base vectors of $\mathscr{Y}$, the following row vectors are linearly independent:

$$\begin{aligned} &c_1, c_1 A, \ldots\ldots\ldots\ldots\ldots, c_1 A^{l_1 - 1}, \\ &c_2, c_2 A, \ldots\ldots\ldots\ldots, c_2 A^{l_2 - 1}, \\ &\vdots \\ &c_{p-1}, c_{p-1} A, \ldots, c_{p-1} A^{l_{p-1} - 1}, \\ &c_p, c_p A \ldots, c_p A^{l_p - 1}. \end{aligned} \tag{2.12}$$

Hence, if we take $\kappa_1 \geqslant \kappa_2 \geqslant \cdots \geqslant \kappa_p$ such that $-1 \leqslant \kappa_i \leqslant l_{i-1}$, then W is surjective and the dynamic order of the observer (2.10) is

$$n_a = n - p - \sum_{i=1}^{p} \kappa_i .$$

Remark 2.2. Note that the 'full order' dynamic observer may be recovered from this result by taking $\kappa_i = -1$ for all i, i.e. by not allowing direct feedthrough of any of the components of y (or equivalently: $W = 0$). Also note that the 'reduced order' observer may be recovered from the above result by taking $\kappa_i = 0$ for all i, i.e. by allowing direct feedthrough of $y(t)$ only and not of any of its derivatives (equivalently: $W = C$).

Remark 2.3. If we take $\kappa_i \geqslant l_i - 1$, we obtain $n_a = 0$. In this case the observer (2.10) degenerates into a *PD-observer*, or following [5], into an *instantaneously acting observer*.

3. Almost controllability subspace covers

In this section we will generalize the concepts of cover and cover-index as introduced in [4]. Let (A, B) be a controllable pair, where $B: \mathscr{U} \to \mathscr{X}$ is an injective linear map and $\mathscr{U} := \mathbb{R}^m$. Denote $\mathscr{B} := \operatorname{im} B$. Let $\mathscr{L} \subset \mathscr{X}$. Recall that an (A, B)-invariant subspace $\mathscr{V}$ of $\mathscr{X}$ is called a *cover* for $\mathscr{L}$ if $\mathscr{L} \subset \mathscr{B} + \mathscr{V}$. Consider the following generalization: Let $\mathscr{R}_a$ be an almost controllability subspace rel. (A, B). (see [8]). Then an (A, B)-invariant subspace $\mathscr{V}$ of $\mathscr{X}$ will be called an *$\mathscr{R}_a$-cover* for $\mathscr{L}$ if $\mathscr{L} \subset \mathscr{R}_a + \mathscr{V}$. Note that, in our terminology, a cover in the sense of [4] would be called a $\mathscr{B}$-cover. We will define the $\mathscr{R}_a$-cover index of $\mathscr{L}$ to be the smallest integer $\nu \geqslant 0$ such that the following holds: for every symmetric set Λ of ν complex numbers with the property

$$\Lambda \neq \emptyset \;\Rightarrow\; \Lambda \cap \mathbb{R} \neq \emptyset,$$

there exists an $\mathscr{R}_a$-cover $\mathscr{V}$ for $\mathscr{L}$ and a map F: $\mathscr{X} \to \mathscr{U}$ such that

$$\dim \mathscr{V} = \nu, \qquad (A + BF)\mathscr{V} \subset \mathscr{V}$$

and

$$\sigma(A + BF \mid \mathscr{V}) = \Lambda .$$

From [6], Theorem 5.1, we immediately obtain that every subspace $\mathscr{L} \subset \mathscr{X}$ has an $\mathscr{R}_a$-cover of dimension $n - \dim \mathscr{R}_a$ and thus that the $\mathscr{R}_a$-cover index ν of $\mathscr{L}$ is well-defined and satisfies

$$0 \leqslant \nu \leqslant n - \dim \mathscr{R}_a .$$

As already noted in [4], the problem of computing the $\mathscr{B}$-cover index and the corresponding $\mathscr{B}$-covers for an arbitrary subspace $\mathscr{L}$ is unsolved. However, for the case that $\dim \mathscr{L} = 1$, a complete solution was described in [4]. In this note we will extend the latter result to the problem of computing the $\mathscr{R}_a$-cover index and corresponding $\mathscr{R}_a$-covers for $\mathscr{L}$, in the case that the almost controllability subspace $\mathscr{R}_a$ is assumed to be equal to $\mathscr{K}_k$, where

$$\mathscr{K}_k := \begin{cases} \mathscr{B} + A\mathscr{B} + \cdots + A^{k-1}\mathscr{B}, & k \geqslant 0, \\ \{0\}, & k = 0. \end{cases} \tag{3.1}$$

Recall from [4] that controllability subspaces $\mathscr{R}_i$ $(1 \leqslant i \leqslant m)$ exist such that

$$\mathscr{X} = \mathscr{R}_1 \oplus \mathscr{R}_2 \oplus \cdots \oplus \mathscr{R}_m$$

and such that $\mathscr{R}_i \cap \mathscr{B} = \ell_i$ is one-dimensional. Writing $\mu_i := \dim \mathscr{R}_i$, the integers μ_i are the controllability indices. We will assume $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_m$. Moreover, there is a map $F_0: \mathscr{X} \to \mathscr{U}$ such that, with $A_0 := A + BF_0$,

$$A_0 \mathscr{R}_i \subset \mathscr{R}_i \quad \text{and} \quad \mathscr{R}_i = \ell_i \oplus A_0 \ell_i \oplus \cdots \oplus A_0^{\mu_i - 1} \ell_i .$$

Let $b_i \in \mathscr{B}$ be such that $\operatorname{span}\{b_i\} = \ell_i$. Define

$$\mathscr{R}_{m+1} := \{0\} \quad \text{and} \quad \mu_{m+1} := 0.$$

Now take $k \geqslant 0$ and for any subspace $\mathscr{L} \subset \mathscr{X}$ define

$$l := \max\left\{ i : 1 \leqslant i \leqslant m + 1 \;\&\; \mathscr{L} \subset \mathscr{K}_k + \sum_{j=1}^{m+1} \mathscr{R}_j \right\}. \tag{3.2}$$

Then we have the following lemma, the proof of which will be given along the lines of [4], Lemma 3.2:

Lemma 3.1. *Assume that* $\dim \mathscr{L} = 1$. *Suppose* Λ *is a symmetric set of* $\max\{0, \mu_l - k\}$ *complex numbers with the property*

$$\Lambda \neq \emptyset \Rightarrow \Lambda \cap \mathbb{R} \neq \emptyset.$$

Then there is an $\mathscr{H}_k$*-cover* $\mathscr{V}$ *for* $\mathscr{L}$ *and a map* F: $\mathscr{X} \to \mathscr{U}$ *such that*

$$\dim \mathscr{V} = \max\{0, \mu_l - k\},$$

$$(A + BF)\mathscr{V} \subset \mathscr{V} \quad \text{and} \quad \sigma(A + BF \mid \mathscr{V}) = \Lambda.$$

Proof. If $\mu_l \leqslant k$, then clearly

$$\mathscr{R}_l \oplus \mathscr{R}_{l+1} \oplus \cdots \oplus \mathscr{R}_m \subset \mathscr{H}_k,$$

so in this case $\mathscr{V} = \{0\}$ is an $\mathscr{H}_k$-cover for $\mathscr{L}$. Assume now that $\mu_l > k$. Apply [6], Theorem 5.1 to the system $(A_0|\mathscr{R}_l, b_l)$ to obtain $\mathscr{V}_l \subset \mathscr{R}_l$ and $f_l: \mathscr{X} \to \mathbb{R}$ with

$$(A_0 + b_l f_l)\mathscr{V}_l \subset \mathscr{V}_l,$$

$$\mathscr{E}_l \oplus A_0 \mathscr{E}_l \oplus \cdots \oplus A_0^{k-1} \mathscr{E}_l \oplus \mathscr{V}_l = \mathscr{R}_l$$

and

$$(A_0 + b_l f_l | \mathscr{V}_l) = \Lambda.$$

It can be arranged that

$$f_l | \mathscr{R}_i = 0 \quad (i \neq l).$$

For $l < i \leqslant m$, if $\mu_i \leqslant k$, take $f_i = 0$ and $\mathscr{V}_i = \{0\}$. If $\mu_i > k$, take a symmetric subset $\Lambda_i \subset \Lambda$ of $\mu_i - k$ complex numbers and apply [6], Theorem 5.1, to $(A_0|\mathscr{R}_i, b_i)$ to obtain $\mathscr{V}_i \subset \mathscr{R}_i$ and f_i: $\mathscr{X} \to \mathbb{R}$ with

$$(A_0 + b_i f_i)\mathscr{V}_i \subset \mathscr{V}_i,$$

$$\mathscr{E}_i \oplus A_0 \mathscr{E}_i \oplus \cdots \oplus A_0^{k-1} \mathscr{E}_i \oplus \mathscr{V}_i = \mathscr{R}_i,$$

$$\sigma(A_0 \oplus b_i f_i | \mathscr{V}_i) = \Lambda_i$$

and

$$f_i | \mathscr{R}_j = 0 \quad (i \neq j).$$

(Note that it is always possible to choose a symmetric subset $\Lambda_i \subset \Lambda$ for $\mu_i - k$ odd or even because of the assumption $\mathscr{L} \cap \mathbb{R} \neq \emptyset$.) Let $\hat{F}$: $\mathscr{X} \to \mathscr{U}$ be such that

$$B\hat{F} = b_l f_l \oplus \cdots \oplus b_m f_m.$$

Define

$$\hat{\mathscr{V}} := \mathscr{V}_l \oplus \cdots \oplus \mathscr{V}_m.$$

Obviously, $(A_0 + B\hat{F})\hat{\mathscr{V}} \subset \hat{\mathscr{V}}$ and

$$\mathscr{R}_l \oplus \mathscr{R}_{l+1} \oplus \cdots \oplus \mathscr{R}_m \subset \hat{\mathscr{V}} + \mathscr{H}_k. \tag{3.3}$$

Since $A_0|\mathscr{R}_l$ is cyclic, it can be seen that $(A_0 + B\hat{F})|\mathscr{V}_l$ is cyclic. Let $\pi(s)$ be the characteristic polynomial of $(A_0 + B\hat{F}) | \mathscr{V}_l$. Since $\Lambda_i \subset \Lambda$, the minimal polynomial of $(A_0 + B\hat{F})|\hat{\mathscr{V}}$ must be equal to $\pi(s)$. Let $z \in \mathscr{X}$ be a vector such that $\mathscr{L} = \operatorname{span}\{z\}$. From (3.2) and (3.3.), $z - h \in \hat{\mathscr{V}}$ for some $h \in \mathscr{H}_k$. Define

$$\mathscr{V}_0 := \langle A_0 + B\hat{F} | \operatorname{span}\{z - h\} \rangle.$$

Then $(A + B\hat{F})|\mathscr{V}_0$ is cyclic and hence, since $\mathscr{V}_0 \subset \hat{\mathscr{V}}$, its characteristic polynomial must divide $\pi(s)$. Now, if $\dim \mathscr{V}_0 = \mu_l - k$, let $\mathscr{V} := \mathscr{V}_0$ and $F := F_0 + \hat{F}$. If $\dim \mathscr{V}_0 < \mu_l - k$, apply [4], Lemma 3.1, to obtain $\mathscr{V}$ and F with the desired properties. □

The following theorem now generalizes the result from [4]:

Theorem 3.2. *Let* $k \geqslant 0$ *and assume that* $\dim \mathscr{L} = 1$. *Then the* $\mathscr{H}_k$*-cover index* $\nu(k)$ *of* $\mathscr{L}$ *is given by*

$$\nu(k) = \max\{0, \mu_l - k\}. \tag{3.4}$$

Proof. From the previous lemma, the $\mathscr{H}_k$-cover index $\nu(k)$ of $\mathscr{L}$ satisfies

$$\nu(k) \leqslant \max\{0, \mu_l - k\}.$$

The proof of the reverse inequality can be given by adapting the proof of the corresponding result from [4], p. 99, using ingredients similar to those in the proof of Lemma 3.1 above. □

4. Minimal order PID-observers

In this section we will introduce formal definitions of the concepts of PID-observer and minimality of order. Starting from these definitions, we will explain in which sense the observers (1.2), (1.7) and (2.10) are minimal. Finally we will dualize the results from Section 3 to establish the existence of minimal order PID-observers for a single linear functional of the state. This result will generalize the well known results from [4] (see also [7], p. 77).

Again, consider the system (1.1) and assume that a second output equation

$$z = Dx \tag{4.1}$$

is given. Here, $z \in \mathscr{Z} := \mathbb{R}^q$ will be interpreted as the variable to be identified by the observer. In the following, let $\mathscr{N}_a$ be an almost complementary observability subspace (rel. (C, A)). Then a (C, A)-invariant subspace $\mathscr{S} \subset \mathscr{X}$ will be called a *PID-observer (rel. $\mathscr{N}_a$) for Dx* if

$$\mathscr{S} \cap \mathscr{N}_a \subset \ker D. \tag{4.2}$$

The *$\mathscr{N}_a$-observer index* of Dx is the smallest integer $\nu \geqslant 0$ such that the following holds: For every symmetric set Λ of ν complex numbers with the property

$$\Lambda \neq \emptyset \;\Rightarrow\; \Lambda \cap \mathbb{R} \neq \emptyset,$$

there exists a PID-observer $\mathscr{S}$ (rel. $\mathscr{N}_a$) for Dx and a map $K: \mathscr{Y} \to \mathscr{X}$ such that

$$\dim \mathscr{X}(\operatorname{mod} \mathscr{S}) = \nu, \qquad (A + KC)\mathscr{S} \subset \mathscr{S}$$

and

$$\sigma((A + KC) \operatorname{mod} \mathscr{S}) = \Lambda.$$

A PID-observer $\mathscr{S}$ (rel. $\mathscr{N}_a$) for Dx will be said to have minimal order if $\dim \mathscr{X}(\operatorname{mod} \mathscr{S})$ is equal to the $\mathscr{N}_a$-observer index of Dx. The above definitions generalize definitions by Wonham and Morse [4]. In fact, their definitions of observer and observer index can be recovered from the above ones by taking $\mathscr{N}_a = \ker C$. Note the duality between the notions of PID-observer and cover. A little thought reveals that a (C, A)-invariant subspace $\mathscr{S}$ is a PID-observer (rel. $\mathscr{N}_a$) for Dx if and only if $\mathscr{S}^{\perp}$ is an $\mathscr{N}_a^{\perp}$-cover for $\operatorname{im} D^{\mathrm{T}}$ (rel. $(A^{\mathrm{T}}, C^{\mathrm{T}})$). Moreover, the $\mathscr{N}_a$-observer index of Dx is equal to the $\mathscr{N}_a^{\perp}$-cover index of $\operatorname{im} D^{\mathrm{T}}$ (see also [5]).

We will now explain how the above formal definition of PID-observer yields a 'real' PID-observer, i.e. a system with differentiators identifying $z(t)$. For this, specify integers $\kappa_1 \geqslant \kappa_2 \geqslant \cdots \geqslant \kappa_p$ as in Section 2 and take $\mathscr{N}_a = \ker W$, where W is given by (2.3). Now, let $\mathscr{S}$ be a PID-observer (rel. $\mathscr{N}_a$) for Dx and suppose $\mathscr{S}$ has minimal order. Let $\nu = \dim \mathscr{X}(\operatorname{mod} \mathscr{S})$ be the $\mathscr{N}_a$-observer index of Dx. Let $V: \mathscr{X} \to \mathbb{R}^{\nu}$ be such that $\ker V = \mathscr{S}$. Let Λ be a symmetric set of ν complex numbers with $\Lambda = \emptyset$ if $\nu = 0$ and $\Lambda \cap \mathbb{R} \neq \emptyset$ if $\nu > 0$. Clearly, from (4.2) we have

$$\ker\begin{pmatrix} W \\ V \end{pmatrix} \subset \ker D.$$

Therefore, maps

$$M = (M_0, M_1, \ldots, M_{\kappa_1}): \mathscr{Y}_0 \oplus \mathscr{Y}_1 \oplus \cdots \oplus \mathscr{Y}_{\kappa_1} \to \mathscr{Z}$$

and

$$N: \mathbb{R}^{\nu} \to \mathscr{Z}$$

exist such that

$$MW + NV = D.$$

Moreover, there are maps $K: \mathscr{Y} \to \mathscr{X}$ and $J: \mathbb{R}^{\nu} \to \mathbb{R}^{\nu}$ with

$$V(A + KC) = JV \quad \text{and} \quad \sigma(J) = \Lambda.$$

Now consider the system

$$\Sigma_{\text{obs}}: \quad \dot{w} = Jw - VKy, \tag{4.3a}$$

$$\hat{z} = Nw + M_0 L_0 y + M_1 L_1 y^{(1)} + \cdots + M_{\kappa_1} L_{\kappa_1} y^{(\kappa_1)}. \tag{4.3b}$$

Then $e := w - Vx$ satisfies $\dot{e} = Jw$ and $\hat{z} - z = Ne$. Hence, taking Λ in the open left half complex plane yields

$$\hat{z}(t) - z(t) \to 0 \quad (t \to \infty).$$

We see that (4.3) defines a system that identifies $z(t)$. Σ_{obs} will also be called a PID-observer. This PID-observer Σ_{obs} has minimal order in the sense that its dynamic order is equal to the $\mathscr{N}_a$-observer index of Dx. In general, the problem of finding the $\mathscr{N}_a$-cover index of Dx, being dual to the cover problem, is very difficult. We do have a result which treats the case that $D = I$:

Proposition 4.1. *Let $\mathscr{N}_a$ be an almost complementary observability subspace (rel. (C, A)). Then the $\mathscr{N}_a$-observer index of X $(= Ix)$ is equal to $\dim \mathscr{N}_a$.*

Proof. From Theorem 2.1, for every symmetric set Λ with $\Lambda \cap \mathbb{R} \neq \emptyset$, there is a PID-observer $\mathscr{S}$ (rel. $\mathscr{N}_a$) and $K: \mathscr{Y} \to \mathscr{X}$ with

$$\dim \mathscr{X}(\operatorname{mod} \mathscr{S}) = \dim \mathscr{N}_a$$

and

$$\sigma((A + KC) \operatorname{mod} \mathscr{S}) = \Lambda.$$

Hence the $\mathscr{N}_a$-observer index ν of Dx satisfies

$\nu \leqslant \dim \mathcal{N}_a$. On the other hand, every PID-observer $\mathcal{S}$(rel. $\mathcal{N}_a$) for x must satisfy $\mathcal{N}_a \cap \mathcal{S} = \{0\}$, so also $\nu \geqslant \dim \mathcal{N}_a$. □

From the above proposition, note that the PID-observer Σ_{obs} as defined by (2.10) has minimal order, in the sense that its dynamic order n_a ($= \dim \mathcal{N}_a$) is equal to the $\mathcal{N}_a$-observer index of x. As a special case of this, we obtain that the 'full order' observer (1.2) *has minimal order* (its dynamic order $n_a = n$ is equal to the $\mathcal{N}_a$-observer index of x with $\mathcal{N}_a = \mathcal{X}$). As another special case we obtain that the 'reduced order' observer has minimal order (its dynamic order $n_a = n - p$ is equal to the $\mathcal{N}_a$-observer index of x with $\mathcal{N}_a = \ker C$).

Remark 4.2. There is still another sense in which the 'reduced order' observer (1.7) is minimal. Consider the set $\Omega := \{\mathcal{L} \mid \mathcal{L}$ is a subspace of $\mathcal{X}$ and $\ker C \subset \mathcal{L}\}$. Then every $\mathcal{L} \in \Omega$ is an almost complementary observability subspace. Hence, we may apply Theorem 2.1 to each element $\mathcal{L}$ in Ω to obtain, for each symmetric set Λ of $\dim \mathcal{L}$ complex numbers, a (C, A)-invariant subspace $\mathcal{S}$ and a map $K: \mathcal{Y} \to \mathcal{X}$ such that

$$\mathcal{L} \oplus \mathcal{S} = \mathcal{X}, \qquad (A + KC)\mathcal{S} \subset \mathcal{S}$$

and

$$\sigma((A + KC) \bmod \mathcal{S}) = \Lambda.$$

Corresponding to $\mathcal{L} \in \Omega$ there is a matrix L_0 such that $\mathcal{L} = \ker W$, with $W = L_0 C$. This yields an observer

$$\Sigma_{\text{obs}}: \quad \dot{w} = Jw - VKy,$$
$$\hat{x} = Nw + M_0 L_0 y$$

with $\sigma(J) = \Lambda$, which only uses $L_0 y(t)$ for direct feedthrough. Therefore, the set Ω parametrizes the set of all PI-observers Σ_{obs} for the state x, in the sense that each $\mathcal{L} \in \Omega$ yields a PI-observer with dynamic order $\dim \mathcal{L} \geqslant n - p$. Taking $\mathcal{L} = \ker C$ yields the 'reduced order' observer, which has dynamic order $n - p$. Hence we may state that *the 'reduced order' observer has minimal dynamic order over the set of all PI-observers for the state x*.

Finally, we will dualize the results from Section 3 to establish the existence of minimal PID-observers for a single linear functional of the state. In (4.1), assume that $\mathcal{Z} = \mathbb{R}$ and to stress this write $D = d$, where d is a linear functional on $\mathcal{X}$. We will assume that the entire observation $y(t)$, together with all its derivatives up to the order $k - 1$ may be used for direct feedthrough. This corresponds to taking $\kappa_1 = \kappa_2 = \cdots = \kappa_p = k - 1$. Here, $k = 0$ means that no direct feedthrough is allowed. The mapping W corresponding to this choice is $W = W_k$, where

$$W_k := \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{pmatrix}, \qquad W_0 := 0. \tag{4.4}$$

Note that for all i, $L_i = I$ and $\mathcal{Y}_i = \mathcal{Y}$. Denote $\mathcal{N}_k = \ker W_k$. As noted before, the $\mathcal{N}_k$-observer index of dx is equal to the $\mathcal{N}_k^{\perp}$-cover index of $\operatorname{im} d^{\mathrm{T}}$ (rel. the pair $(A^{\mathrm{T}}, C^{\mathrm{T}})$). The latter integer $\nu(k)$ can be found using Theorem 3.2. It follows immediately that for each symmetric set Λ of $\nu(k)$ complex numbers with $\Lambda = \emptyset$ if $\nu(k) = 0$ and $\Lambda \cap \mathbb{R} \neq \emptyset$ if $\nu(k) > 0$, a minimal order PID-observer $\mathcal{S}$ (rel. $\mathcal{N}_k$) for dx exists, and a map $K: \mathcal{Y} \to \mathcal{X}$ such that

$$(A + KC)\mathcal{S} \subset \mathcal{S}, \qquad \sigma((A + KC) \bmod \mathcal{S}) = \Lambda.$$

This leads to a PID-observer Σ_{obs} for $z(t)$:

$$\Sigma_{\text{obs}}: \quad \dot{w} = Jw - VKy, \tag{4.5a}$$
$$\hat{z} = nw + m_0 y + m_1 y^{(1)} + \cdots + m_{k-1} y^{(k-1)}, \tag{4.5b}$$

with $\sigma(J) = \Lambda$, of dynamic order $\nu(k)$. Here, n and m_i are linear functionals on $\mathcal{W}$ and $\mathcal{Y}$ respectively. The observer Σ_{obs} has minimal order in the sense that its dynamic order is equal to the $\mathcal{N}_k$-observer index of dx. Note that the original result by Wonham and Morse [4] can be recovered from the above by taking $k = 1$.

In particular, for a given $d: \mathcal{X} \to \mathbb{R}$ it is possible to find a PID-observer Σ_{obs} for $z = dx$, using $y(t)$, $y^{(1)}(t), \ldots, y^{(k-1)}(t)$ for direct feedthrough, of dynamic order $\max\{0, l_1 - k\}$. Here, l_1 is the largest observability index of the pair (C, A). It is always possible to find an I-observer for $z = dx$ (no direct feedthrough at all) of dynamic order l_1.

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